EN5101 Digital Control Systems

Controllability and Observability

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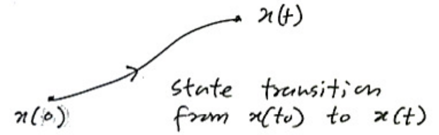
Controllability

Controllability : A system is controllable if the state could be transfered from an initial state $x(t_0)$ to a desired state $x(t)$ by means of an unconstrained signal in a finite interval
[Note: a check whether all the states are affected by the control input]



System $\dot{x} = Ax + Bu$, where x is $n\times1$, A is $n\times n$, B is $n\times1$ and u is a scalar

$$x(t) = 0 = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$e^{At}x(0) = -\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$x(0) = -\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

from Cayley - Hamilton Theorem

$$e^{-A\tau} = \sum_{k=0}^{n-1}\alpha_k(\tau)A^k \quad \text{then,}$$

$$x(0) = -\sum_{k=0}^{n-1}A^kB\int_0^t \alpha_k(\tau)u(\tau)\,d\tau$$

$$= -\underbrace{\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B\right]}_{C\text{ - Controllability matrix}}\begin{bmatrix}\int_0^t \alpha_0(\tau)u(\tau)d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau)u(\tau)d\tau\end{bmatrix}$$

There should be n linearly independent rows in C in order to satisfy controllability of any $x(0)$.
Full state controllability $\Rightarrow$ C full rank.

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{B}u$$

The controllability matrix $C = [B \,\vdots\, AB] = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \Rightarrow$ singular

The rank of C is less than 2 (controllability matrix is not full-rank). The system is not controllable.

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

MatLab
$C = ctrb(A, B), rank(C)$

The controllability matrix $C = [B \,\vdots\, AB] = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ nonsingular

The controllability matrix is full-rank (rank $C = 2$)

The system is controllable.

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u \qquad C = [B \vdots AB] = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \text{ singular}$$

The state x_1 is controllable as $\dot{x}_1 = x_1 + u$ (u is there) but the state x_2 is uncontrollable as $\dot{x}_2 = -x_2$

Even though state $\dot{x}_1 = x_1$ is unstable it could be controlled by the control input $\dot{x}_1 = x_1 + u$

Even though state $\dot{x}_2 = -x_2$ is not controllable (as there is no contribution from u there). The state is naturally stable $\dot{x}_2 = -x_2$.

There fore the system is stabilizable. The system can be resolved into two parts

$$\dot{x}_1 = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + u \qquad \text{controllable unstable part}$$

$$\dot{x}_2 = [0 \quad -1]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad \text{uncontrollable, but stable part.}$$

Observability

A system is completely observable if the state $x(t)$ can be determined from the observation $y(t)$ over a finite time.

[checks whether the output is contributed by all states].

given the system

$$\dot{x} = Ax \qquad \text{neglect } Bu \text{ term, which doesn't affect system observability}$$

$$y = Cx$$

then,

$$y(t) = Cx(t)$$
$$\dot{y}(t) = C\dot{x}(t) = CAx(t)$$
$$\vdots$$
$$y^{n-1}(t) = Cx^{n-1}(t) = CA^{n-1}x(t)$$

$y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $C_{m\times n}$, $A_{n\times n}$

$$\underbrace{\begin{bmatrix}y(t)\\ \dot{y}(t)\\ \vdots\\ y^{n-1}(t)\end{bmatrix}_{mn\times 1}}_{\text{output and its derivatives}} = \underbrace{\begin{bmatrix}C_{m\times n}\\ CA_{m\times n}\\ \vdots\\ CA^{n-1}_{m\times n}\end{bmatrix}_{mn\times n}}_{\text{observability matrix}} x(t)_{n\times 1}$$

The state $x(t)$ can be uniquely determined if the observability matrix is full-rank

Example
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The observability matrix $\begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ is of rank 2

The System is Observable.

MatLab
$O = obsv(A,C), rank(O)$